NOTES AND QUERIES.

We have received the following from Mr. William Orchard, F.I.A.:—

Petersburg Problem.—This is the name of a celebrated problem in the Theory of Probabilities, which has given rise to much discussion.

The problem is this:—A coin is thrown until head appears: if it appear at the first time, A is to pay B £2; if at the second, £4; if at the nth throw, £2^n. What should B pay to A before the commencement of the game for his expectation of gain by it?

The probability of head appearing the first time is $\frac{1}{2}$; the gain if it appear is £2; therefore the value of B's expectation upon this throw is £1; and, generally, the value of his expectation upon the nth throw is £1; for that the nth throw may occur, head must not have appeared in the first $(n-1)$ throws, the probability of which is $\frac{1}{2^{n-1}}$; that it will then appear at the following throw is $\frac{1}{2}$, and the consequent gain £2^n: therefore his expectation is $\frac{1}{2^{n-1}} \cdot \frac{1}{2} \cdot 2^n = 1$. If the game is to be continued until head appears, however long it may be deferred, B's expectation of gain is infinite; for an infinite number of throws are possible, and must be allowed for in the calculation.

Although such is the strict mathematical conclusion, no one, not even a mathematician, would give much for the expectation of gain which the game offers. Daniel Bernoulli, in discussing the problem in the *Petersburg Memoirs*, whence its name, invented a theory of *moral* expectation, distinguished from *mathematical* expectation by the consideration that the value of a sum of money to an individual depends upon the amount of his previous fortune. By means of this theory,—for which, and its application to the present problem, I refer to the *Essay on Probability, L. U. K.*, De Morgan's Treatise on Probability in the *Encyclopædia Metropolitana*, and Galloway's Treatise from the *Encyclopædia Britannica*,—he deduced a finite and very small value for B's moral expectation, depending upon the amount of his fortune, but not at all upon that of A.

Poisson has, in his *Recherches sur la Probabilité des Jugements*, given a very happy solution of the problem, without having recourse to the theory of moral expectation. As this solution has not appeared in any English treatise, I lay a sketch of it before your readers, referring to Poisson's own work for more detail and generality.

Let A's fortune be £2^n, then if head appear in the first n throws, A will be able to pay the whole loss: B's expectation upon these n throws is £n. But if head be not thrown until after n throws, however much B may gain by the conditions of the game, he can only receive from A the whole amount of his fortune, that is, £2^n. From this cause B's expectation on all the throws after the nth is reduced to—

$$2^n \left\{ \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots\dots \right\} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\dots = 1;$$

and therefore the value of B's entire expectation is $(n+1)$ pounds.

Thus, if A's fortune were so large as $2^{20} = 1{,}048{,}576$ pounds, the sum which B would be justified in paying him for the possible gains under the conditions of the problem would be but £21.